

Plane Symmetric Gravitational Collapse

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Abstract

In this paper, we derive the general formulation by considering two arbitrary plane symmetric spacetimes using Israel's method. As an example, we apply this formulation to known plane symmetric spacetimes. We take the Taub's static metric in the interior region whereas Kasner's non-static metric in the exterior region. It is shown that the plane collapses in some cases whereas it expands in some other cases.

Keywords : Gravitational Collapse

1 Introduction

An important issue in relativistic astrophysics and theory of General Relativity (GR) is to determine the end state of gravitational collapse. It has many interesting applications in astrophysics where the formation of compact stellar objects such as white dwarf and neutron star are formed by a period of gravitational collapse. The singularity theorems of Hawking and Penrose [1] show that if a trapped surface forms during the collapse of a compact object made out of physically reasonable matter, such a collapse will develop a spacetime singularity. A spacetime singularity is defined as the region in the spacetime where the evolution of geodesics will be incomplete.

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The singularity theorems do not provide information about the visibility of a spacetime singularity. A spacetime singularity is called a naked singularity if it is visible to an observer. A spacetime singularity which is not visible is called a black hole. Penrose [2] proposed that the spacetime singularities must be hidden inside an event horizon. This means that the gravitational collapse must end as a black hole. This is called the cosmic censorship conjecture (CCC). The CCC is also one of the most important open problems in gravitational physics today. Since many physical applications of black hole and several other areas in GR depend on CCC, hence a detailed study of dynamically developing gravitational collapse models is necessary to obtain a correct form of the CCC. To study gravitational collapse in the framework of GR, it is necessary to consider the appropriate geometry of interior and exterior regions and determine proper junction conditions which allow the matching of these regions.

The pioneering work on gravitational collapse was first started by Oppenheimer and Snyder [3]. They studied collapse of dust by considering a static Schwarzschild in exterior and Friedmann like solution in interior. Since then, many people [4-14] have extended the above study of collapse by taking an appropriate geometry of interior and exterior regions. Recently, the effect of a positive cosmological constant on spherically symmetric collapse with perfect fluid has been investigated [15] by considering the matching conditions between static exterior and non-static interior spacetimes. All these studies are restricted to spherical gravitational collapse. On the other hand, the study about non-spherical collapse is limited than spherical collapse. Shapiro and Teukolsky [16] studied numerically the problem of a dust spheroid and found that a black hole could be formed if the spheroid is compact enough. Otherwise, the end state of the collapse will be a naked singularity. Barrabs et al. [17] investigated an analytical model of a collapsing convex thin shell and showed that apparent horizons are not formed in some cases. These results were soon generalized to more general cases [18]. Since then, these studies have attracted attention to non-spherical collapse.

Villas da Rocha et al. [19] worked on the self-similar gravitational collapse of perfect fluid using Israel's method [20]. Applying the same analysis, Pereira and Wang [21] studied the gravitational collapse of cylindrical shells made of counter rotating dust particles. They derived general formulas by considering two arbitrary cylindrically symmetric spacetimes and applied these by taking two known cylindrically symmetric spacetimes. They concluded that in some cases the shell collapses and in some cases it expands.

The same authors [22] have also discussed the dynamics of expanding and collapsing cylindrically symmetric fields with lightlike wave-fronts. There is a large body of literature [23- 26] available on gravitational collapse which show keen interest to investigate this issue by using Israel's formalism. In this paper, we extend the work done by Pereira and Wang [21] for cylindrically symmetric spacetimes to plane symmetric spacetimes. The paper is outlined as follows. In next section, we derive general formulae by considering two arbitrary plane symmetric spacetimes. Section 3 provides application of these formulae to known plane symmetric spacetimes. We consider the Taub's static metric in the interior region whereas Kasner's non-static metric in the exterior region. Finally, in section 4, we discuss and conclude the results.

2 General Formalism

We consider a timelike $3D$ hypersurface Σ , which divides $4D$ spacetime into two regions interior and exterior spacetimes, denoted by V^+ and V^- respectively. The region V^- is described by the metric

$$ds_-^2 = f^-(t, z)dt^2 - g^-(t, z)(dx^2 + dy^2) - h^-(t, z)dz^2, \quad (1)$$

where $\{\chi^{-\mu}\} \equiv \{t, x, y, z\}$, $(\mu = 0, 1, 2, 3)$ are the usual Cartesian coordinates. The metric for the region V^+ is given by

$$ds_+^2 = f^+(T, Z)dT^2 - g^+(T, Z)(dx^2 + dy^2) - h^+(T, Z)dZ^2, \quad (2)$$

where $\{\chi^{+\mu}\} \equiv \{T, x, y, Z\}$, $(\mu = 0, 1, 2, 3)$ is another set of the Cartesian coordinates.

According to the junction condition [20,27], it is assumed that the interior and the exterior spacetimes are the same on the hypersurface Σ which can be expressed as

$$(ds_-^2)_\Sigma = (ds_+^2)_\Sigma = ds_\Sigma^2. \quad (3)$$

The extrinsic curvature tensor K_{ab}^\pm to hypersurface Σ is defined as [20]

$$K_{ab}^\pm = n_\sigma^\pm \left(\frac{\partial^2 x_\pm^\sigma}{\partial \xi^a \partial \xi^b} + \Gamma_{\mu\nu}^\sigma \frac{\partial x_\pm^\mu}{\partial \xi^a} \frac{\partial x_\pm^\nu}{\partial \xi^b} \right), \quad (\sigma, \mu, \nu = 0, 1, 2, 3), \quad (4)$$

where $(a, b = 0, 1, 2)$. Here the Christoffel symbols $\Gamma_{\mu\nu}^\sigma$ are calculated from the interior or exterior metrics (1) or (2), n_μ^\pm are the components of outward

unit normals to Σ in the coordinates $\chi^\pm\sigma$. The equations of hypersurface Σ in the coordinates $\chi^\pm\sigma$ are written as

$$k_-(z, t) = z - z_0(t) = 0, \quad (5)$$

$$k_+(Z, T) = Z - Z_0(T) = 0. \quad (6)$$

Using Eq.(5) in (1), the metric on Σ takes the form

$$(ds_-^2)_\Sigma = [f^-(t, z_0(t)) - h^-(t, z_0(t))z_0'^2(t)]dt^2 - g^-(t, z_0(t))(dx^2 + dy^2). \quad (7)$$

Similarly, Eqs.(2) and (6) yield

$$(ds_+^2)_\Sigma = [f^+(T, Z_0(T)) - h^+(T, Z_0(T))Z_0'^2(T)]dT^2 - g^+(T, Z_0(T))(dx^2 + dy^2), \quad (8)$$

where prime means ordinary differentiation with respect to the indicated argument. The metric on the hypersurface Σ is given by

$$(ds^2)_\Sigma = \gamma_{ab}d\xi^a d\xi^b = d\tau^2 - g(\tau)(dx^2 + dy^2). \quad (9)$$

From the junction condition (3), it follows that

$$\begin{aligned} d\tau &= [f^-(t, z_0(t)) - h^-(t, z_0(t))z_0'^2(t)]^{\frac{1}{2}}dt, \\ &= [f^+(T, Z_0(T)) - h^+(T, Z_0(T))Z_0'^2(T)]^{\frac{1}{2}}dT, \end{aligned} \quad (10)$$

$$g(\tau) = g^-(t, z_0(t)) = g^+(T, Z_0(T)). \quad (11)$$

The outward unit normals in V^- and V^+ can be evaluated by using Eqs.(5) and (6)

$$n_\mu^+ = \left[\frac{f^+ h^+}{f^+ - h^+ Z_0'^2(T)} \right]^{\frac{1}{2}} (-Z_0'(T), 0, 0, 1), \quad (12)$$

$$n_\mu^- = \left[\frac{f^- h^-}{f^- - h^- z_0'^2(t)} \right]^{\frac{1}{2}} (-z_0'(t), 0, 0, 1). \quad (13)$$

The components of the extrinsic curvature K_{ab}^\pm are

$$\begin{aligned} K_{\tau\tau}^+ &= -\frac{(f^+ h^+)^{\frac{1}{2}}}{2[f^+ - h^+ Z_0'^2(T)]^{\frac{3}{2}}} \left\{ -\frac{f_{,Z}^+}{h^+} + \left(\frac{f_{,T}^+}{f^+} - 2\frac{h_{,T}^+}{h^+} \right) Z_0'(T) \right. \\ &\quad \left. + \left(2\frac{f_{,Z}^+}{f^+} - \frac{h_{,Z}^+}{h^+} \right) Z_0'^2(T) + \frac{h_{,T}^+}{f^+} Z_0'^3(T) - 2Z_0''(T) \right\}, \\ K_{xx}^+ &= -\frac{1}{2} \left[\frac{f^+ h^+}{f^+ - h^+ Z_0'^2(T)} \right]^{\frac{1}{2}} \left\{ \frac{g_{,Z}^+}{h^+} + \frac{g_{,T}^+}{f^+} Z_0'(T) \right\} = K_{yy}^+, \end{aligned} \quad (14)$$

where comma denotes partial derivative and K_{ab}^- can be obtained from the above expressions by the replacement

$$f^+, g^+, h^+, Z_0(T), T, Z \rightarrow f^-, g^-, h^-, z_0(t), t, z. \quad (15)$$

The surface energy-momentum tensor, τ_{ab} , is defined as [20,21]

$$\tau_{ab} = \frac{1}{\kappa} \{ [K_{ab}]^- - \gamma_{ab} [K]^- \}, \quad (16)$$

where κ is the gravitational constant and

$$[K_{ab}]^- = K_{ab}^+ - K_{ab}^-, \quad [K]^- = \gamma_{ab} [K_{ab}]^-. \quad (17)$$

Using Eq.(14) and the corresponding expressions for K_{ab}^- into Eq.(16), τ_{ab} can be expressed in the form

$$\tau_{ab} = \rho w_a w_b + p(x_a x_b + y_a y_b), \quad (a, b = \tau, x, y), \quad (18)$$

where ρ is the surface energy density, p is the tangential pressure provided that they satisfy some energy conditions [28] and w_a , x_a , y_a are unit vectors defined on the surface given by

$$w_a = \delta_a^\tau, \quad x_a = h^{\frac{1}{2}}(\tau) \delta_a^x, \quad y_a = h^{\frac{1}{2}}(\tau) \delta_a^y. \quad (19)$$

Here ρ and p turn out to be

$$\rho = \frac{2}{\kappa g(\tau)} [K_{xx}]^-, \quad p = \frac{1}{\kappa} [[K_{\tau\tau}]^- - \frac{[K_{xx}]^-}{g(\tau)}]. \quad (20)$$

3 Plane Symmetric Gravitational Collapse

In this section, we shall apply the general results developed in the previous section by considering plane symmetric spacetimes. The metric for interior region (Taub's static metric) is given by [29]

$$ds_-^2 = z^{-\frac{1}{2}}(dt^2 - dz^2) - z(dx^2 + dy^2), \quad z > 0. \quad (21)$$

For exterior spacetime, we take the non-static (Kasner's) metric given by [29]

$$ds_+^2 = T^{-\frac{1}{2}}(dT^2 - dZ^2) - T(dx^2 + dy^2), \quad T > 0. \quad (22)$$

Both interior and exterior spacetimes are vacuum, i.e., energy-momentum for both spacetimes is zero. The physical importance to consider these (Taub and Kasner) metrics is that they satisfy energy conditions [28]. Using Eqs.(21) and (22), the junction conditions (10) and (11) take the form

$$d\tau = [1 - z_0'^2(t)]^{\frac{1}{2}} z_0^{-\frac{1}{4}} dt = [1 - Z_0'^2(T)]^{\frac{1}{2}} T^{-\frac{1}{4}} dT, \quad (23)$$

$$z_0(t) = T. \quad (24)$$

From Eqs.(23) and (24), it turn out that

$$\left(\frac{dT}{dt}\right)^2 = \frac{1}{\Delta^2} \equiv [2 - Z_0'^2]^{-1} \quad (25)$$

and Eqs.(24) and (25) yield

$$z_0''(t) = \frac{d^2T}{dt^2} = \frac{Z_0'Z_0''}{\Delta^4}. \quad (26)$$

Using Eq.(14) and the corresponding expressions for K_{ab}^- into Eq.(20) and considering Eqs.(24)-(26), we obtain

$$\rho = \frac{2}{\kappa T^{\frac{3}{4}}(1 - Z_0'^2)^{\frac{1}{2}}}(\Delta - Z_0'), \quad (27)$$

$$p = \frac{\Delta - Z_0'}{4\kappa T^{\frac{3}{4}}\Delta(1 - Z_0'^2)^{\frac{3}{2}}}\{4TZ_0'' - \Delta(1 - Z_0'^2)\}. \quad (28)$$

In order to see the minimal effects of the plane on the collapse, we shall take $p = 0$ in Eq.(28). This leads to the following two cases

$$(A) : \quad \Delta - Z_0' = 0, \quad (29)$$

$$or \quad (B) : \quad 4TZ_0'' - \Delta(1 - Z_0'^2) = 0, \quad (30)$$

where

$$\Delta = \pm\sqrt{2 - Z_0'^2}. \quad (31)$$

It is mentioned here that equation of state i.e., $\rho = ap$ can be used to see pressure effects on the gravitational collapse. Since, we want to see the minimum effects of the shell on the collapse therefore, we take $p = 0$. The case(A) is trivial because surface energy density becomes zero from Eq.(27) and thus we leave it. Case(B) is further divided into two subcases:

$$\begin{aligned} \text{B(i)} \quad & 4TZ_0'' - \sqrt{2 - Z_0'^2}(1 - Z_0'^2) = 0, \\ \text{B(ii)} \quad & 4TZ_0'' + \sqrt{2 - Z_0'^2}(1 - Z_0'^2) = 0. \end{aligned}$$

In the following, we solve these two cases separately.

For the case B(i), integration of the equation

$$4TZ_0'' - \sqrt{2 - Z_0'^2}(1 - Z_0'^2) = 0 \quad (32)$$

yields

$$Z_0'(T) = \pm \frac{\sqrt{2} \tanh(\frac{\ln(T)}{4})}{\sqrt{1 + \tanh(\frac{\ln(T)}{4})^2}}, \quad (33)$$

where constant of integration is taken zero for the sake of simplicity. It is mentioned here that the " + " and " - " signs correspond to an expanding and collapsing planes respectively.

Expanding plane

Integrating Eq.(33) with " + " sign, we obtain

$$Z_0(T) = \frac{(-2 + \sqrt{T} - 2T + T^{\frac{3}{2}} - \sqrt{1+T} \sinh^{-1}[\sqrt{T}])}{(T^{\frac{1}{4}} \sqrt{\frac{1+T}{\sqrt{T}}})}, \quad (34)$$

where integration constant is taken zero to avoid complicated situation. Eqs.(33) and (34) yield respectively

$$\begin{aligned} Z_0(T) &= \begin{cases} 0.126418 & , \quad T = 7, \\ +\infty & , \quad T = +\infty, \end{cases} \\ Z_0'(T) &= \begin{cases} 0.581861 & , \quad T = 7, \\ 1 & , \quad T = +\infty \end{cases} \end{aligned} \quad (35)$$

indicating that the plane is expanding. The plane starts to expand at time $T = 7$, where it has displacement 0.126418, velocity 0.581861 and positive acceleration. The expansion ends at $T = +\infty$, where the plane has displacement $+\infty$, velocity unity and zero acceleration. It can be seen from Eq.(27) that the energy density decreases from finite value to zero in this interval.

Collapsing plane

Integration of Eq.(33) with "-" sign yields

$$Z_0(T) = -\frac{(-2 + \sqrt{T} - 2T + T^{\frac{3}{2}} - \sqrt{1+T} \sinh^{-1}[\sqrt{T}])}{(T^{\frac{1}{4}} \sqrt{\frac{1+T}{\sqrt{T}}})}, \quad (36)$$

where integration constant is taken zero. From Eqs.(33) and (36), we obtain collapsing plane

$$\begin{aligned} Z_0(T) &= \begin{cases} 2.29559 & , \quad T = 1, \\ 0.439096 & , \quad T = 6, \end{cases} \\ Z'_0(T) &= \begin{cases} 0 & , \quad T = 1, \\ -0.547856 & , \quad T = 6. \end{cases} \end{aligned} \quad (37)$$

The plane starts to collapse at time $T = 1$, where it has radial displacement 2.29559, zero velocity and positive acceleration and ends at $T = 6$, where the plane has displacement 0.439096 and negative velocity. It is to be noticed that energy density remains finite in this interval i.e., collapse does end as a singularity. This situation might be change if one replaces interior and exterior metrics which are physical i.e., which satisfy energy conditions. It is mentioned here that plane collapses and expands along Z -direction only because Z is not a radial coordinate.

The case B(ii) gives similar results as the case B(i).

4 Conclusion

This paper is an extension of the previous studies of spherical and cylindrical gravitational collapse to plane symmetric gravitational collapse using Israel's method. First of all, we have developed a general formalism for two arbitrary plane symmetric spacetimes in terms of the metric coefficients and their first derivatives. Then, as an example, we have applied this formulation to known plane symmetric spacetimes. We have taken the Taub's static metric in the interior region whereas Kasner's non-static metric in the exterior region. We have taken these metrics because they satisfy energy conditions [28]. It is found that the plane collapses in some cases whereas it expands in some other cases.

There arise two main cases A and B according to Eqs.(29) and (30) respectively. The case A is trivial. For the case B , there is expanding as well as collapsing process. It is mentioned here that in one case of B(i), the plane collapses for all $T > 0$ and in other case of B(i), it expands for all $T > 0$. But physically, the plane collapses in the interval $1 \leq T \leq 6$ and expands in the interval $7 \leq T < \infty$. Notice that the collapsing interval is less than the expanding interval. Since we have taken vacuum spacetimes in the interior and exterior regions. This result is consistent with that the cosmological constant, i.e., vacuum energy density slows down the collapsing process [30].

We would like to mention here that the collapse does not end as a singularity because energy density remains finite at all times in both cases. The physical implication of this work is that the general formalism can be applied by considering different physical plane symmetric spacetimes (which satisfy energy conditions) to get interesting result.

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